Gravitational Radiation from a Naked Singularity

— *Odd-Parity Perturbation* —

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It has been suggested that a naked singularity is a good candidate for a strong gravitational wave burster. A naked singularity occurs in the generic collapse of an inhomogeneous dust ball. We study the odd-parity mode of gravitational waves from a naked singularity of the Lemaître-Tolman-Bondi spacetime. The wave equation for gravitational waves is solved by numerical integration using the single null coordinate. It is found that the naked singularity is not a strong source of the odd-parity gravitational radiation, although the metric perturbation grows in the central region. Therefore, the Cauchy horizon in this spacetime should be marginally stable with respect to odd-parity perturbations.

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I. INTRODUCTION

The study involved in the attempt to detect gravitational waves represents one of the most exciting field in modern physics. The detection will provide a decisive test of general relativity and open a new window for astrophysical observation. Unfortunately, in spite a great deal of effort, no one has yet succeeded in the detection of the gravitational waves. However ground-based laser interferometers, such as LIGO, [1] VIRGO, [2] TAMA [3] and GEO600, [4] are being constructed and will soon enter a stage of data taking. Strong sources of gravitational waves should be regions where gravity is general relativistic and where the velocities of bulk motion are near the speed of light. Many researchers have theoretically investigated coalescing and colliding black holes and neutron stars as candidates for detection. In addition, gravitational waves emitted during stellar collapse to black holes have been studied by perturbative calculations of a spherically symmetric background. [5,6] Because of the existence of event horizon, the emitted gravitational waves are dominated by the quasi-normal modes of the Schwarzschild black hole in the asymptotic region. Some other realistic candidates exist, e.g., supernovae, rapidly rotating neutron stars, relic gravitational waves of the early universe, stochastic gravitational waves from cosmic strings, and so on.

In this paper we attempt to investigate whether a naked singularity, if such exists, is a strong source of gravitational radiation, and, moreover, to understand the dynamics and observational meaning of naked singularity formation. Several researchers have shown that the final fate of gravitational collapse is not always a singularity covered by an event horizon. For example, in the Lemaître-Tolman-Bondi (LTB) spacetime, [7,8] a naked shell-focusing singularity appears from generic initial data for spherically symmetric configurations of the rest mass density and a specific energy of the dust fluid. [9–12] The initial functions in the most general expandable form have been considered. [13] The matter content in such a spacetime may satisfy even the dominant energy condition. In this case with a small disturbance of the spacetime, very short wavelength gravitational waves, which are created in the high density region around a singularity, may propagate to the observer outside the dust cloud because of the absence of an event horizon. If this is true, extremely high energy phenomena which cannot be realized by any high energy experiment on Earth can be observed. In this case information regarding the physics of so-called 'quantum gravity' may be obtained. Also, these waves may be so intense that they destroy the Cauchy horizon. In this paper the generation of gravitational waves during the collapse of spherical dust ball with small rotational motion is considered.

Nakamura, Shibata and Nakao [14] have suggested that a naked singularity may emit considerable gravitational wave radiation. This was proposed from the estimate of gravitational radiation from a spindle-like naked singularity.

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They modeled the spindle-like naked singularity formation in gravitational collapse by the Newtonian prolate dust collapse for a sequence of general relativistic, momentarily static initial data. It should be noted that the system they considered is different from that considered in this article and that their result is controversial. There are numerical analyses that both support and do not support the results of Nakamura, Shibata and Nakao for prolate collapse [15] and for cylindrical collapse. [16,17]

Due to the non-linear nature of the problem, it is difficult to analytically solve the Einstein equation. Therefore, numerical methods will provide the final tool. However, its singular behavior makes accurate numerical analysis very difficult at some stage. In this article, we investigate odd-parity linear gravitational waves from the collapse of an inhomogeneous spherically symmetric dust cloud. Even for the linearized Einstein equation, we must perform the numerical integration. However, in contrast to the numerical simulation of the full Einstein equation, high precision is guaranteed for the numerical integration of the linearized Einstein equation, even in the region with extremely large spacetime curvature. Furthermore, the linear stability of known examples of naked singularity formation is necessary as a first step to understand the general dynamics near the "naked singularity formation."

Recently, Iguchi, Nakao and Harada [18] (INH) studied odd-parity metric perturbations around a naked singularity in the LTB spacetime. In INH, it was found that the propagation of odd-parity gravitational waves is not affected by collapse of a dust cloud, even if there appears a central naked singularity. When we consider the generation of gravitational waves from the dust collapse, we should analyze the perturbations including their matter part. Here we investigate the evolution equation of the odd-parity quadrupole mode for metric and matter perturbations. This matter perturbation relates to small rotational motion, and it produces gravitational waves during the collapse. We follow the derivation of the evolution equations of the perturbations and the numerical method to integrate these equations of INH. We investigate the time evolutions of the gauge invariant metric variables at the symmetric center, where a naked singularity appears, and at a constant circumferential radius R. We show that the gauge invariant variable diverges only at the center and it does not propagate to the outside.

As we know, the LTB spacetime is one candidate as a counterexample of the cosmic censorship hypothesis (CCH), which was introduced by Penrose. [19,20] The CCH is a very helpful assumption, because various theorems on the properties of black holes were proved under this assumption. Here the precise formulation and validity of cosmic censorship is not our concern. We only consider the situation in which the extremely high-density and large-curvature region can be seen by some observer from the gravitational collapse. Such a situation can be regarded as a naked singularity in a practical sense since we are not yet able to predict phenomena beyond the Planck scale. However, here it should be noted that even from this practical point of view, the stability of the Cauchy horizon with respect to gravitational-wave perturbations still has an important physical meaning. Since in the system considered here, the mode propagating with the speed of light is the gravitational waves only, the instability of the Cauchy horizon implies that the gravitational wave introduces an extremely large spacetime curvature along the null hypersurface near the Cauchy horizon from the region near the central singularity. Hence, if the Cauchy horizon in this system is unstable, naked singularity formation of this type might be a strong source of gravitational radiation.

Here we comment on the problem motivated in INH, i.e., "nakedness of the naked singularity." Odd-parity matter perturbations are produced by rotational motion of the dust cloud. Therefore the growth of the matter perturbation would cause the centrifugal force to dominate the radial motion of the cloud. Hence, an angular momentum bounce would occur. If this occurs at the center, the central singularity will disappear. This situation seems to be inevitable for the dipole mode of odd-parity matter perturbations. In the case of a spherically symmetric system composed of counter-rotating particles, the angular-momentum bounce can actually prevent the naked singularity formation. [21] However, for a system without spherical symmetry, it is still an open question how the non-linear asphericity works in the final stage of singularity formation with rotational matter perturbations because of the difficulty of both the analytic and numerical treatment. Further, if the initial matter perturbation is sufficiently small, it might still be possible that the radius of the spacetime curvature becomes the Planck length. In this case, the behavior of other modes of perturbations is crucial to understand the classical dynamics in the region of the Planck scale. As will be shown below, since there is no gravitational wave of the dipole mode and the quadrupole mode generated in the dust cloud does not propagate to the outside, the Cauchy horizon is marginally stable against odd-parity perturbations originating in the aspherical rotational motion of matter.

The paper is organized as follows: In §II, the basic equations are developed; in §III, the numerical results are presented; these results are discussed in §IV; in §V, we summarize our results. We adopt the geometrized units, in which $c = G = 1$. The signature of the metric tensor and sign convention of the Riemann tensor follow Ref. [22].

II. BASIC EQUATIONS

We consider the evolution of odd-parity perturbations of the LTB spacetime up to linear order. The background spacetime and perturbation method are described in INH. [18] In this section we briefly review them.

Using the synchronous coordinate system, the line element of the background LTB spacetime can be expressed in the form

$$d\bar{s}^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu \equiv -dt^2 + A^2(t,r)dr^2 + R^2(t,r)(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.1}$$

The energy-momentum tensor for the dust fluid is

$$\bar{T}^{\mu\nu} = \bar{\rho}(t,r)\bar{u}^\mu \bar{u}^\nu, \tag{2.2}$$

where $\bar{\rho}(t,r)$ is the rest mass density and $\bar{u}^\mu$ is the 4-velocity of the dust fluid. In the synchronous coordinate system, the unit vector field normal to the spacelike hypersurfaces is a geodesic, and there is a freedom of which timelike geodesic field is adopted as the hypersurface unit normal. Using this freedom, we can always set $\bar{u}^\mu = \delta_0^\mu$, since the 4-velocity of the spherically symmetric dust fluid is tangent to an irrotational timelike geodesic field.

Then the Einstein equations and the equation of motion for the dust fluid reduce to the simple equations

$$A = \frac{\partial_r R}{\sqrt{1+f(r)}}, \tag{2.3}$$

$$\bar{\rho}(t,r) = \frac{1}{8\pi}\frac{1}{R^2 \partial_r R}\frac{dF(r)}{dr}, \tag{2.4}$$

$$(\partial_t R)^2 - \frac{F(r)}{R} = f(r), \tag{2.5}$$

where $f(r)$ and $F(r)$ are arbitrary functions of the radial coordinate, r. From Eq. (2.4), $F(r)$ is related to the Misner-Sharp mass function, [23] $m(r)$, of the dust cloud in the manner

$$m(r) = 4\pi \int_0^{R(t,r)} \bar{\rho}(t,r)R^2 dR = 4\pi \int_0^r \bar{\rho}(t,r)R^2 \partial_r R dr = \frac{F(r)}{2}. \tag{2.6}$$

Hence Eq. (2.5) might be regarded as the energy equation per unit mass. This means that the other arbitrary function, $f(r)$, is recognized as the specific energy of the dust fluid. The motion of the dust cloud is completely specified by the function $F(r)$ (or equivalently, the initial distribution of the rest mass density, $\bar{\rho}$) and the specific energy, $f(r)$. When we restrict our calculation to the case that the symmetric center, $r=0$, is initially regular, the central shell focusing singularity is naked if and only if $\partial_r^2 \bar{\rho}|_{r=0} < 0$ is initially satisfied for the marginally bound collapse, $f(r)=0$. [24,25] For a collapse that is not marginally bound, there exists a similar condition as an inequality for a value depending on the functional forms of $F(r)$ and $f(r)$. [11,24,25]

Let us now derive the perturbation equations. The perturbed metric tensor is expressed in the form

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{2.7}$$

where $\bar{g}_{\mu\nu}$ is a metric tensor of a spherically symmetric background spacetime and $h_{\mu\nu}$ is a perturbation. The energy-momentum tensor is written in the form

$$T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu}, \tag{2.8}$$

where $\bar{T}_{\mu\nu}$ is a background quantity and $\delta T_{\mu\nu}$ is a perturbation. By virtue of the spherical symmetry of the background spacetime, $\bar{T}_{\mu\nu}$ can be expressed in the form

$$\bar{T}_{\mu\nu}dx^\mu dx^\nu = \bar{T}_{ab}dx^a dx^b + \frac{1}{2}\bar{T}_B{}^B R^2(t,r)d\Omega^2, \tag{2.9}$$

where the subscripts and superscripts $a, b, \cdots$ represent t and r, while $A, B, \cdots$ represent θ and ϕ. The odd-parity perturbations of $h_{\mu\nu}$ and $\delta T_{\mu\nu}$ are expressed in the form

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & h_0(t,r)\Phi_l^m{}_B \\ & 0 & h_1(t,r)\Phi_l^m{}_B \\ \text{sym} & & h_2(t,r)\chi_l^m{}_{AB} \end{pmatrix}, \tag{2.10}$$

$$\delta T_{\mu\nu} = \begin{pmatrix} 0 & 0 & t_0(t,r)\Phi^m_{l\ B} \\ & 0 & t_1(t,r)\Phi^m_{l\ B} \\ \text{sym} & & t_2(t,r)\chi^m_{l\ AB} \end{pmatrix}, \tag{2.11}$$

where $\Phi^m_{l\ B}$ and $\chi^m_{l\ AB}$ are odd-parity vector and tensor harmonics associated with the spherical symmetry of the background spacetime. [26] We set all the arbitrary constants in the definitions of harmonics to unity. We then introduce the gauge-invariant variables defined by Gerlach and Sengupta. [27] The metric variables are given by

$$k_a = h_a - \frac{1}{2}R^2 \partial_a \left(\frac{h_2}{R^2}\right). \tag{2.12}$$

The matter variables are given by the combinations

$$L_a = t_a - \frac{1}{2}\bar{T}_B^{\ B}h_a, \tag{2.13}$$

$$L = t_2 - \frac{1}{2}\bar{T}_B^{\ B}h_2. \tag{2.14}$$

In the LTB case, the odd-parity gauge-invariant matter variables become

$$L_0 = \bar{\rho}(t,r)U(t,r) \qquad \text{and} \qquad L_1 = L = 0, \tag{2.15}$$

where $U(t,r)$ represents the perturbation of the 4-velocity as $\delta u_\mu = (0,0,U(t,r)\Phi^m_{l\ B})$. The evolution equation for the matter variable (Eq. (3·19) in INH),

$$\partial_t \left(AR^2 L_0\right) = 0, \tag{2.16}$$

is easily integrated, and we obtain

$$L_0 = \frac{1}{AR^2}\frac{dJ(r)}{dr}, \tag{2.17}$$

where $J(r)$ is an arbitrary function depending only on r. From Eqs. (2.4), (2.15), and (2.17), we obtain the relation

$$U(t,r) = 8\pi\sqrt{1+f(r)}\frac{dJ(r)/dr}{dF(r)/dr} \tag{2.18}$$

$$\equiv U(r), \tag{2.19}$$

so $U(t,r)$ is independent of the time coordinate t. We introduce a gauge-invariant variable for the metric as

$$\psi_s \equiv \frac{1}{A}\left[\partial_t \left(\frac{k_1}{R^2}\right) - \partial_r \left(\frac{k_0}{R^2}\right)\right]. \tag{2.20}$$

The metric perturbation variables, k_0 and k_1, are reconstructed from the linearized Einstein equations,

$$\partial_r \left(R^4\psi_s\right) + A\left(l-1\right)\left(l+2\right)k_0 = 16\pi AR^2 L_0, \tag{2.21}$$

$$\partial_t \left(R^4\psi_s\right) + \frac{1}{A}\left(l-1\right)\left(l+2\right)k_1 = 0, \tag{2.22}$$

by substituting ψ_s. It was shown in INH that ψ_s is closely connected to the tetrad components of the magnetic part of the Weyl tensor. These components diverge where and only where ψ_s diverges. From the linearized Einstein equations we obtain the linearized evolution equation for the odd-parity perturbation as

$$\partial_t \left(\frac{A}{R^2}\partial_t \left(R^4\psi_s\right)\right) - \partial_r \left(\frac{1}{AR^2}\partial_r \left(R^4\psi_s\right)\right) + (l-1)(l+2)A\psi_s$$

$$= -16\pi\partial_r \left(\frac{1}{AR^2}\frac{dJ}{dr}\right). \tag{2.23}$$

The regularity conditions at the center are also considered in INH. The result for the matter perturbation L_0 is given by

$$L_0 \longrightarrow L_c(t)r^{l+1} + O(r^{l+3}). \tag{2.24}$$

Therefore the matter perturbation variables vanish at the regular center independent of the value of l. The metric variable ψ_s behaves near the center as

$$\psi_s \longrightarrow \psi_{sc}(t)r^{l-2} + O(r^l) \qquad \text{for } l \geq 2, \tag{2.25}$$
$$\psi_s \longrightarrow \psi_{sc}(t)r + O(r^3) \qquad \text{for } l = 1. \tag{2.26}$$

From the above equations, it is shown that only the quadrupole mode, $l = 2$, of ψ_s does not vanish at the regular center.

III. RESULTS

We numerically solve the wave equation (2.23) in the case of marginally bound collapse, $f(r) = 0$, and the quadrupole mode, $l = 2$. We follow numerical techniques employed in INH to integrate the time evolution of the perturbations. The numerical code is essentially the same as that in INH, which is tested by comparison with the analytic solution for the Minkowski background.

By virtue of the relation $f(r) = 0$, we can easily integrate Eq. (2.5) and obtain

$$R(t,r) = \left(\frac{9F}{4}\right)^{1/3} [t_0(r) - t]^{2/3}, \tag{3.1}$$

where $t_0(r)$ is an arbitrary function of r. The naked singularity formation time is $t_0 = t_0(0)$. Using the freedom for the scaling of r, we choose $R(0,r) = r$. This scaling of r corresponds to the following choice of $t_0(r)$:

$$t_0(r) = \frac{2}{3\sqrt{F}}r^{3/2}. \tag{3.2}$$

Here note that, from Eq. (2.3), the background metric variable, A, is equal to $\partial_r R$. Then, the wave equation (2.23) becomes

$$\frac{\partial^2 \psi_s}{\partial t^2} - \frac{1}{(\partial_r R)^2}\frac{\partial^2 \psi_s}{\partial r^2} = \frac{1}{(\partial_r R)^2}\left(6\frac{\partial_r R}{R} - \frac{\partial_r^2 R}{\partial_r R}\right)\frac{\partial \psi_s}{\partial r} - \left(6\frac{\partial_t R}{R} + \frac{\partial_t \partial_r R}{\partial_r R}\right)\frac{\partial \psi_s}{\partial t}$$
$$-4\left[\left(\frac{\partial_t \partial_r R}{\partial_r R}\right)\frac{\partial_t R}{R} + \frac{1}{2}\left(\frac{\partial_t R}{R}\right)^2\right]\psi_s$$
$$-\frac{16\pi}{(\partial_r R)R^2}\partial_r\left(\frac{r^2 \rho(r)U(r)}{(\partial_r R)R^2}\right), \tag{3.3}$$

where $\rho(r) = \bar{\rho}(0,r)$ is the density profile at $t = 0$. We solve this partial differential equation numerically.

Before getting into the detailed explanation of the numerical techniques, we comment on the behavior of the matter perturbation variable L_0 around a naked singularity on the slice $t = t_0$. The regularity conditions of L_0 and $\bar{\rho}$ determine the behavior of $U(r)$ near the center as

$$U(r) \propto r^{l+1}. \tag{3.4}$$

This property does not change even if a central singularity appears. However, the r dependence of R and A near the center changes at that time. Assuming a rest mass density profile of the form

$$\rho(r) = \rho_0 + \rho_n r^n + \cdots, \tag{3.5}$$

we obtain the relation

$$t_0(r) \propto t_0 + t_n r^n \tag{3.6}$$

from Eqs. (2.4) and (3.2), where n is a positive even integer. After substituting this relation into Eq. (3.1), the lowest order term is absent from the square brackets of it. Then we obtain the behavior of R and A around the central singularity as

$$R(t_0, r) \propto r^{1+\frac{2}{3}n}, \tag{3.7}$$

and

$$A(t_0, r) \propto r^{\frac{2}{3}n} \tag{3.8}$$

on the slice $t = t_0$. As a result, we obtain the r dependence of L_0 around the center when the naked singularity appears as

$$L_0(t_0, r) \propto r^{l-2n+1}. \tag{3.9}$$

For example, if $l = 2$ and $n = 2$, then L_0 is inversely proportional to r and diverges toward the central naked singularity. Therefore the source term of the wave equation is expected to have a large magnitude around the naked singularity. Thus the metric perturbation variable ψ_s as well as matter variable L_0 may diverge toward the naked singularity.

Instead of the (t, r) coordinate system, we introduce a single-null coordinate system, (u, r'), where u is an outgoing null coordinate and chosen so that it agrees with t at the symmetric center and we choose $r' = r$. We perform the numerical integration along two characteristic directions. Therefore we use a double null grid in the numerical calculation. A detailed explanation of these coordinates is given in INH. By using this new coordinate system, (u, r'), Eq. (3.3) is expressed in the form

$$\frac{d\phi_s}{du} = -\frac{\alpha}{R}\left[3\partial_r R + \frac{1}{2}R(\partial_t R)\partial_t\partial_r R - \frac{5}{4}(\partial_t R)^2\partial_r R\right]\psi_s$$
$$-\frac{\alpha}{2}\left[\frac{\partial_r^2 R}{(\partial_r R)^2} - \frac{2}{R}\left(1 - \partial_t R\right)\right]\phi_s - \frac{8\pi\alpha}{R}\partial_r\left(\frac{r^2\rho(r)U(r)}{(\partial_r R)R^2}\right), \tag{3.10}$$

$$\partial_{r'}\psi_s = \frac{1}{R}\phi_s - 3\frac{\partial_r R}{R}\left(1 + \partial_t R\right)\psi_s, \tag{3.11}$$

where the ordinary derivative on the left-hand side of Eq. (3.10) and the partial derivative on the left-hand side of Eq. (3.11) are given by

$$\frac{d}{du} = \partial_u + \frac{dr'}{du}\partial_{r'} = \partial_u - \frac{\alpha}{2\partial_r R}\partial_{r'}$$
$$= \frac{\alpha}{2}\partial_t - \frac{\alpha}{2\partial_r R}\partial_r, \tag{3.12}$$

$$\partial_{r'} = -\frac{(\partial_r u)_t}{(\partial_t u)_r}\partial_t + \partial_r = (\partial_r R)\partial_t + \partial_r, \tag{3.13}$$

respectively. Also, ϕ_s is defined by Eq. (3.11) and α is given by

$$\alpha \equiv \frac{1}{(\partial_t u)_r}. \tag{3.14}$$

We integrate Eq. (3.10) using the scheme of an explicit first order difference equation. We use the trapezoidal rule,

$$\psi_{sj+1} = \psi_{sj} + \frac{\Delta r'}{2}\left((\partial_{r'}\psi_s)_j + (\partial_{r'}\psi_s)_{j+1}\right), \tag{3.15}$$

to integrate Eq. (3.11).

For the boundary condition at the center we demand that ψ_s behaves as $\psi_{sc}(t) + \psi_{s2}(t)r^2$ on a surface of $t = $ const. We numerically realize this condition by two-step interpolation. First the values of ψ_s are derived at two points on the surface of $t = $ const from the interpolation on the slices of $u = $ const. Next, using these two values, the central value of ψ_s is derived from the interpolation on the slice of $t = $ const. Another way to determine the central value of ψ_s is as follows. We first obtain the central value of ϕ_s from Eq. (3.10). From Eq. (3.11) and the boundary conditions, the relation of ψ_s and ϕ_s at the center is given by

$$\phi_s = 3\partial_{r'}R\psi_s. \tag{3.16}$$

Using this relation the central value of ψ_s is obtained. In our numerical analyses the results of these two methods agree well.

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We assume ψ_s vanishes on the initial null hypersurface. Therefore, there exist initial ingoing waves which offset the waves produced by the source term on the initial null hypersurface. In INH, it is confirmed that this type of the initial ingoing waves propagate through the dust cloud without net amplification even when they pass through the cloud just before the appearance of the naked singularity. Therefore those parts of the metric perturbations would not diverge at the center.

We adopt the initial rest mass density profile

$$\rho(r) = \rho_0 \frac{1 + \exp\left(-\frac{1}{2}\frac{r_1}{r_2}\right)}{1 + \exp\left(\frac{r^n - r_1^n}{2r_1^{n-1}r_2}\right)}, \tag{3.17}$$

where ρ_0, r_1 and r_2 are positive constants and n is a positive even integer. As a result the dust fluid spreads all over the space. However, if $r \gg r_1, r_2$, then $\rho(r)$ decreases exponentially, so that the dust cloud is divided into the core part and the envelope which would be considered as the vacuum region essentially. We define a core radius as

$$r_{\mathrm{core}} = r_1 + \frac{r_2}{2}. \tag{3.18}$$

If we set $n = 2$, there appears a central naked singularity. This singularity becomes locally or globally naked depending on the parameters (ρ_0, r_1, r_2). However, if the integer n is greater than 2, the final state of the dust cloud is a black hole independently of the parameters. Then we consider three different density profiles connected with three types of the final state of the dust cloud, globally and locally naked singularities and a black hole. The outgoing null coordinate u is chosen so that it agrees with the proper time at the symmetric center. Therefore, even if the black hole background is considered, we can analyze the inside of the event horizon. Corresponding parameters are given in Table I. Using this density profile, we numerically calculate the total gravitational mass of the dust cloud M. In our calculation we adopt the total mass M as the unit of the variables.

The source term of Eq. (3.10),

$$S(t,r) = -\frac{8\pi\alpha}{R}\partial_r\left(\frac{r^2\rho(r)U(r)}{(\partial_r R)R^2}\right), \tag{3.19}$$

is determined by $U(r)$. As mentioned above, the constraints on the functional form of $U(r)$ are given by the regularity condition of L_0. From Eq. (3.4), $U(r)$ should be proportional to r^{l+1} toward the center. We localize the matter perturbation near the center to diminish the effects of the initial ingoing waves. Therefore we define $U(r)$ such that

$$r^2\rho(r)U(r) = \begin{cases} U_0\left(\frac{r}{r_b}\right)^5\left(1 - \left(\frac{r}{r_b}\right)^2\right)^5 & \text{for } 0 \le r \le r_b, \\ 0 & \text{for } r > r_b, \end{cases} \tag{3.20}$$

where U_0 and r_b are arbitrary constants. In our numerical calculation we chose r_b as $r_{\mathrm{core}}/2$. This choice of r_b has no special meaning, and the results of our numerical calculations are not sensitive to it.

First we observe the behavior of ψ_s at the center. The results are plotted in Fig. 1. The initial oscillations correspond to the initial ingoing waves. After these oscillations, ψ_s grows proportional to $(t_0 - t)^{-\delta}$ for the naked singularity cases near the formation epoch of the naked singularity. For the case of black hole formation, ψ_s exhibits power-law growth in the early part. Later its slope gradually changes but it grows faster than in the case of the naked singularity. For the naked singularity cases, the power-law indices δ are determined by $(t_0 - t)\dot{\psi}_s/\psi_s$ locally. The results are shown in Fig. 2. From this figure we read the final indices as $5/3$ for both naked cases. Therefore the metric perturbations diverge at the central naked singularity.

TABLE I. Parameters of initial density profiles, power law indices, and damped oscillation frequencies.

	final state	ρ_0	r_1	r_2	n	power index	damped oscillation frequency
(a)	globally naked	1×10^{-2}	0.25	0.5	2	5/3	—
(b)	locally naked	1×10^{-1}	0.25	0.5	2	5/3	$0.37+0.089i$
(c)	black hole	2×10^{-2}	2	0.4	4	—	$0.37+0.089i$

We also observe the wave form of ψ_s along the line of a constant circumferential radius outside the dust cloud. The results are shown in Figs. 3–5. Figure 3 displays the wave form of the globally naked case (a), Fig. 4 displays the wave form of the locally naked case (b), and Fig. 5 displays the wave form of the black hole case (c). The initial oscillations correspond to the initial ingoing waves. In the case of a locally naked singularity and black hole formation, damped oscillations dominate the gravitational waves. We read the frequencies and damping rates of these damped oscillations from Figs. 4 and 5 and give them in terms of complex frequencies as $0.37+0.089i$ for locally naked and black hole cases. These agree well with the fundamental quasi-normal frequency of the quadrupole mode ($2M\omega = 0.74734 + 0.17792i$) of a Schwarzschild black hole given by Chandrasekhar and Detweiler. [28] In the globally naked singularity case (a), we did not see this damped oscillation because of the existence of the Cauchy horizon. In all cases the gravitational waves generated by matter perturbations are at most quasi-normal modes of a black hole, which is generated outside the dust cloud. Therefore intense odd-parity gravitational waves would not be produced by the inhomogeneous dust cloud collapse. We should not expect that the central extremely high density region can be observed by this mode of gravitational waves.

We can calculate the radiated power of the gravitational waves and thereby grasp the physical meaning of the gauge-invariant quantities. [5] To relate the perturbation of the metric to the radiated gravitational power, it is useful to specialize to the radiation gauge, in which the tetrad components $h_{(\theta)(\theta)} - h_{(\phi)(\phi)}$ and $h_{(\theta)(\phi)}$ fall off as $O(1/R)$, and all other tetrad components fall off as $O(1/R^2)$ or faster. Note that, in vacuum at large distance, the spherically symmetric background metric is given by the Schwarzschild solution, where hereafter we adopt the Schwarzschild coordinates,

$$ds^2 = -\left(1 - \frac{2M}{R}\right)d\tau^2 + \left(1 - \frac{2M}{R}\right)^{-1} dR^2 + R^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{3.21}$$

The relation between the line elements Eq. (2.1) and Eq. (3.21) is given by the transfer matrix

$$d\tau = \frac{1}{1 - (\partial_t R)^2}(dt + \partial_r R \partial_t R dr), \tag{3.22}$$

$$dR = \partial_t R dt + \partial_r R dr. \tag{3.23}$$

In this gauge, the metric perturbations in Eq. (2.10) behave as

$$h_0, h_1 = O\left(\frac{1}{R}\right), \tag{3.24}$$

$$h_2 = w(\tau - R_*) + O(1), \tag{3.25}$$

where

$$R_* = R + 2M \ln\left(\frac{R}{2M} - 1\right) + \text{const.} \tag{3.26}$$

Then, the gauge-invariant metric perturbations (2.12) are calculated as

$$k_0 = -\frac{1}{2}w^{(1)}R + O(1), \tag{3.27}$$

$$k_1 = \frac{1}{2}w^{(1)}R + O(1), \tag{3.28}$$

where $w^{(1)}$ denotes the first derivative of w with respect to its argument.

In this radiation gauge, the radiated power P per unit solid angle is given by the formula which was derived by Landau and Lifshitz [29] from their stress-energy pseudo-tensor:

$$\frac{dP}{d\Omega} = \frac{R^2}{16\pi}\left[\left(\frac{\partial h_{(\theta)(\phi)}}{\partial\tau}\right)^2 + \frac{1}{4}\left(\frac{\partial h_{(\theta)(\theta)}}{\partial\tau} - \frac{\partial h_{(\phi)(\phi)}}{\partial\tau}\right)^2\right]. \tag{3.29}$$

For the axisymmetric mode, i.e., $m = 0$, the above formula is reduced to

$$\frac{dP}{d\Omega} = \frac{1}{64\pi}(w^{(1)})^2 A_l(\theta), \tag{3.30}$$

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where

$$A_l(\theta) \equiv \frac{2l+1}{4\pi} \sin^4 \theta \left(\frac{d^2 P_l(\cos\theta)}{(d\cos\theta)^2} \right)^2.$$ (3.31)

Then, by using the gauge-invariant quantities and integrating over the all solid angles, the formula for the power of gravitational radiation is obtained in the following form:

$$\frac{dP}{d\Omega} = \frac{1}{16\pi} \frac{k_0^2}{R^2} A_l(\theta) = \frac{1}{16\pi} \frac{k_1^2}{R^2} A_l(\theta),$$ (3.32)

$$P = \frac{1}{16\pi} B_l \frac{k_0^2}{R^2} = \frac{1}{16\pi} B_l \frac{k_1^2}{R^2},$$ (3.33)

where

$$B_l \equiv \frac{(l+2)!}{(l-2)!}.$$ (3.34)

Using Eq. (2.22), the radiated power P of the quadrupole mode is given by

$$P = \frac{3}{32\pi} R'^2 \left[\partial_\tau \left(R^3 \psi_s \right) \right]^2.$$ (3.35)

Figure 6 displays the time evolution of the radiated power P. The radiated power also has a finite value at the Cauchy horizon. The total energy radiated by odd-parity quadrupole gravitational waves during the dust collapse should not diverge.

IV. DISCUSSION

First we consider the behavior of the source term $S(t,r)$ around the naked singularity. From the regularity conditions and Eqs. (3.4), (3.7), and (3.8), the asymptotic behavior of the source term is obtained as

$$S(t,r) \propto r^{l-1}$$ (4.1)

for $t < t_0$ and

$$S(t,r) \propto r^{l-\frac{8}{3}n-1},$$ (4.2)

at $t = t_0$. For example, in the case $l = 2$ and $n = 2$, the souce term behaves on $t = t_0$ as

$$S(t,r) \propto r^{-13/3},$$ (4.3)

and then it diverges at the center. Thus the divergency of ψ_s at the center originates from the source term. To confirm this, we numerically integrate the source term along the ingoing null lines with respect to u and estimate the central value of ϕ_s. We define this 'estimated' value as

$$\Phi_s \equiv \int S(t,r) du.$$ (4.4)

Using Eq. (3.16) we can define the estimated value of ψ_s as

$$\Psi_s \equiv \frac{\Phi_s}{3\partial_r R}.$$ (4.5)

We plot it in Fig. 7 together with the corresponding ψ_s. The estimated value has the same power-law index of ψ_s. We conclude that the behavior of ψ_s is determined by the source term in the dust cloud.

We next consider the stability of the Cauchy horizon. We found that the metric perturbation produced by the source term does not propagate outside the dust cloud, except for quasi-normal ringing. The source term, which controls ψ_s, does not diverge at the Cauchy horizon. Therefore ψ_s should not diverge at the Cauchy horizon and should not destroy it. Then, even if odd-parity perturbations are considered, it will not be the case that the LTB

spacetime loses its character as a counterexample to CCH due to Cauchy horizon instability. Also, it does not seem that such collapse is a strong source of gravitational waves.

In this paper, we have dealt with the marginally bound case. For the case of non-marginally bound collapse, the condition of the appearance of the central naked singularity is slightly different from that in the above case [24,25] and hence there is the possibility that the behavior of ψ_s in this case is different from that in the marginally bound case. However, it is well known that the limiting behavior of the metric as $t \to t_0(r)$ is common for all the cases: [29]

$$R \approx \left(\frac{9F}{4}\right)^{1/3} (t_0 - t)^{2/3}, \quad A \approx \left(\frac{2F}{3}\right)^{1/3} \frac{t_0'}{\sqrt{1+f}} (t_0 - t)^{-1/3}. \tag{4.6}$$

We conjecture that the results of the perturbed analysis for the non-marginal collapse would be similar to the results for the marginal bound.

V. SUMMARY

We have studied the behavior of the odd-parity perturbation in the LTB spacetime including the matter perturbation. For the quadrupole mode, where gravitational waves exist, we have numerically investigated the wave equation. For the case of naked singularity formation, the gauge-invariant metric variable, ψ_s, diverges according to a power law with power index 5/3 at the center. This power index is closely related to the behavior of the matter perturbation around the center. We have also observed ψ_s at a constant circumferential radius. For the globally naked case, we cannot see intense gravitational waves propagated from the center just before the crossing of the Cauchy horizon. For the locally naked case, we have confirmed that there exist quasi-normal oscillations. As a result, we conclude that the type of singularity changes due to the odd-parity perturbation because ψ_s diverges at the center. However, the Cauchy horizon is marginally stable against odd-parity perturbations.

At the final stage of the collapse, the effects of the rotational motion are important and the centrifugal force might dominate the radial motion. If this is true, the central singularity would disappear when an odd-parity matter perturbation is introduced. For the dipole mode, such a situation seems to be inevitable. However, we should note that it is a non-trivial and open question how non-linear asphericity affects the final fate of the singularity-formation process. Further, in the case of initially sufficiently small aspherical perturbations, the radius of spacetime curvature at the center might reach the Planck length, and hence there is still the possibility that the naked singularity is formed there in a practical sense. However, as our present analysis has revealed, since the Cauchy horizon is stable with respect to odd-parity linear perturbations, there is little possibility that this collapse is a strong source of odd-parity gravitational waves.

There remains important related works to be completed. The first problem is to consider the even-parity mode in which the metric and matter perturbations are essentially coupled with each other. We are now investigating this problem. Finally, we should consider the non-linear effects to complete this analysis. This problem will be analyzed in the future.

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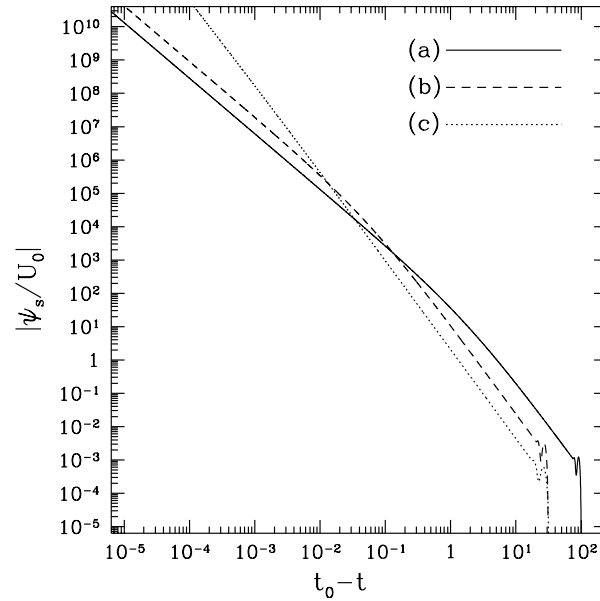[29] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, London, 1975).

FIG. 1. Plots of ψ_s at the center as a function of the time coordinate t. The solid line represents the globally naked case (a), the dashed line represents the locally naked case (b), and the dotted line represents the black hole case (c).
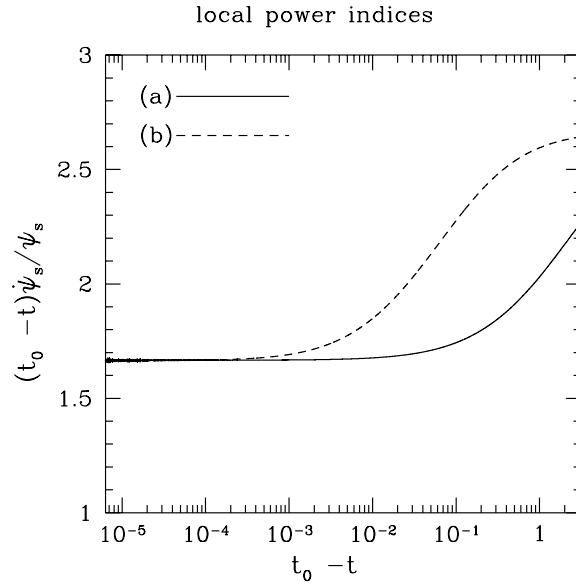
local power indices

FIG. 2. Plots of the local power indices $(t_0 - t)\dot{\psi}_s/\psi_s$. The solid line corresponds to the globally naked case (a), and the dashed line corresponds to the locally naked case (b). Both of them approach a value near 5/3.
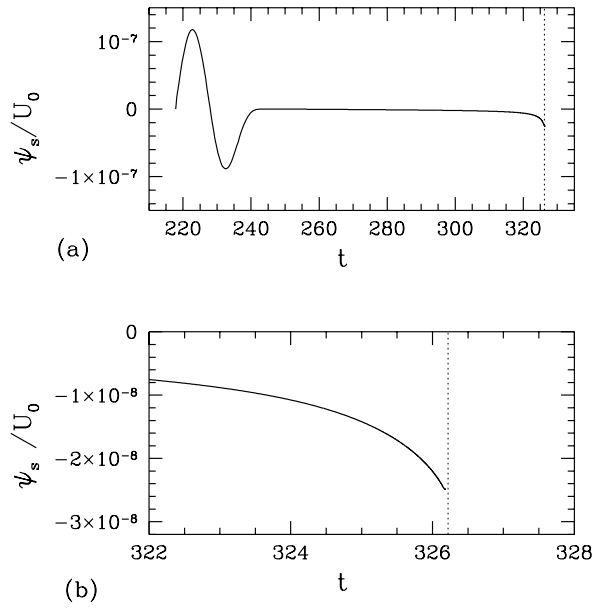

FIG. 3. Plots of ψ_s for the globally naked case (a) at $R = 100$. In (a) the left hand side oscillation originates from the initial ingoing wave. In (b) we magnify the the right-hand edge, which is just before the Cauchy horizon. The dotted lines represent the time at which the observer at $R = 100$ intersects the Cauchy horizon, which is determined by numerical integration of the null geodesic equation from the naked singularity.
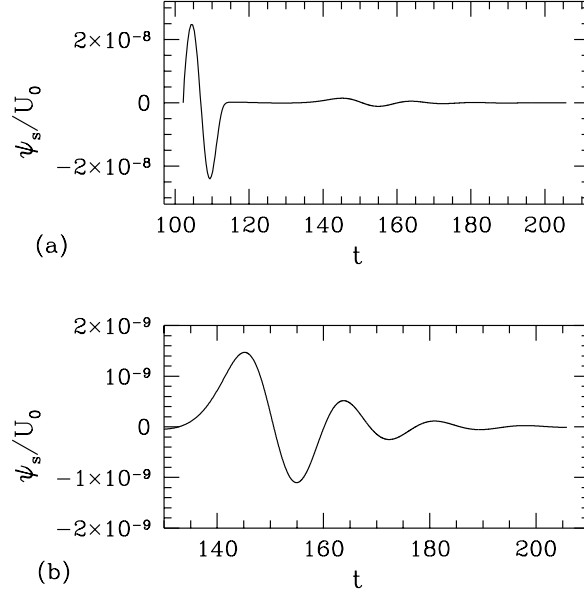
FIG. 4. Plots of ψ_s for the locally naked case (b) at $R = 100$. In (a) the left hand side oscillation originates from the initial ingoing wave. After this oscillation, the damped oscillation dominates, and this part of the wave form is magnified in (b).
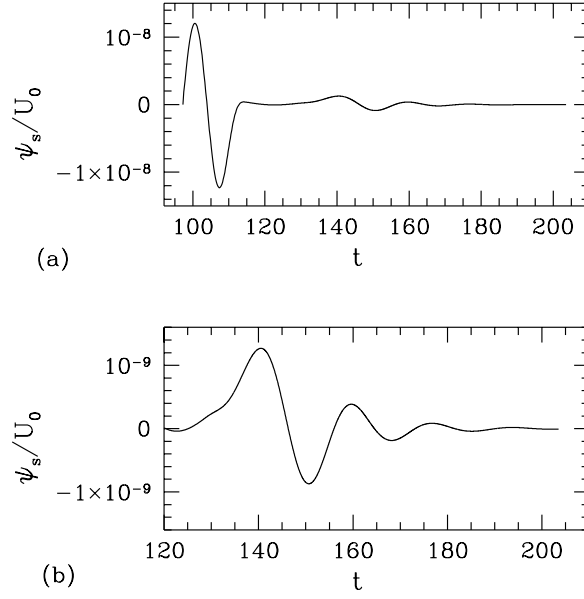

FIG. 5. Plots of ψ_s for the black hole case (c) at $R = 100$. In (a) the left-hand side oscillation originates from the initial ingoing wave. After this oscillation, the damped oscillation dominate, as depicted in (b).
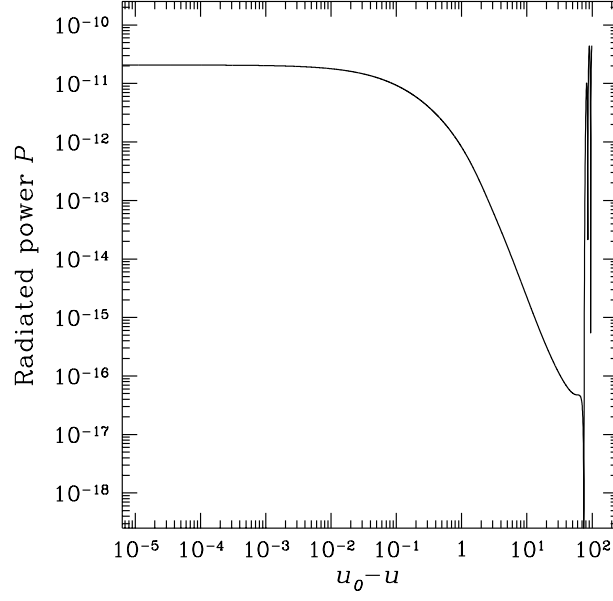
FIG. 6. Plots of the radiated power P for globally naked case (a) at $R = 100$. The horizontal axis is the out-going null coordinate u. At the Cauchy horizon, this coordinate has the value u_0.
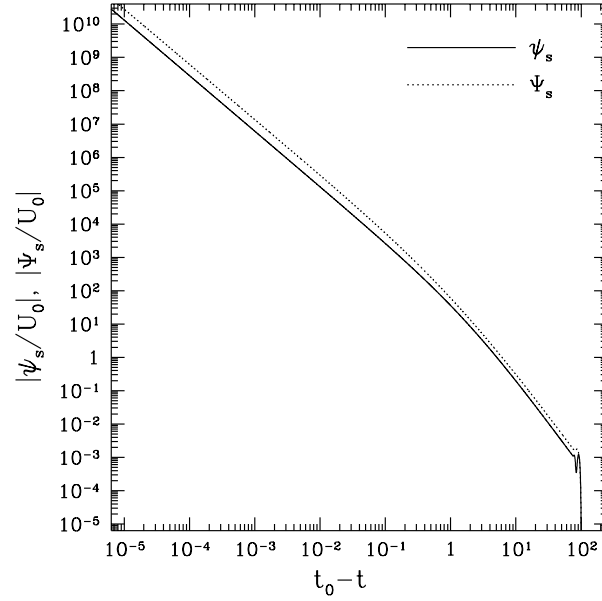

FIG. 7. Plots of ψ_s and the estimated value Ψ_s at the center for the globally naked case (a). The solid line represents the ψ_s, and the dotted line represents the estimated value Ψ_s. Both lines exhibit power-law behavior with power indices 5/3.

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